MATH 548 Homework 3

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Section 7.7

Exercise 1. a)

We make the following conjecture: $f_1 + f_3 + ... + f_{2n-1} = f_{2n}$. We will prove this with induction. The base case is trivial: $(1) + (2) + (5) = (8)$. Our inductive step is $f_{2n} = f_{2n-1} + ... + f_3 + f_1 \Rightarrow f_{2(n+1)} = f_{2n+1} + f_{2n-1} + ... + f_3 + f_1$. Well, from the recurrence relation we know that $f_{2n+2} = f_{2n+1} + f_{2n}$, but by assumption we know that $f_{2n} = f_{2n-1} + ... + f_3 + f_1$, so plugging this into the previous equation we have $f_{2n+2} = f_{2n+1} + f_{2n-1} .... + f_3 + f_1$. And so the proof is done.

Exercise 3 b)

Our base case is simple, we have $3|f_4 = 3$. We must assume $3|f_{4n}$ and show that $3|f_{4(n+1)}$. Well, by the recurrence relation we know that $f_{4(n+1)} = f_{4n+3} + f_{4n+2}$, which we can expand (once again by the recurrence relation) to be $f_{4n+4} = (f_{4n+2} + f_{4n+1}) + (f_{4n+1} + f_{4n})$ which we may once again expand to be $f_{4n+4} = ((f_{4n} + f_{4n+1}) + f_{4n+1}) + (f_{4n+1} + f_{4n})$ which we may rewrite as $f_{4n+4} = 3(f_{4n+1}) + 2(f_n)$, and because we know that f_n is divisible by 3, we know that 2 times f_n must also be divisible by three, and we know that the $3(f_{4n+1})$ term must also be divisible by three because three times any integer is divisible by three, and because the sum of two values that are both divisible by three is also divisible by three, we know that the lefthand side must also be divisible by there and we are done.

Exercise 4

Starting with the recurrence relation $a_n = a_{n-1} + a_{n-2}$, we may rewrite as $a_n = (a_{n-2} + a_{n-3}) + (a_{n-3} + a_{n-4})$ which can again be rewritten as $a_n = ((a_{n-3} + a_{n-4}) + (a_{n-4} + a_{n-5})) + ((a_{n-4} + a_{n-5}) + a_{n-4})$, which can once again be rewritten as $a_n = (((a_{n-4} + a_{n-5}) + a_{n-4}) + (a_{n-4} + a_{n-5})) + ((a_{n-4} + a_{n-5}) + a_{n-4})$ which can be combined to be $a_n = 5(a_{n-4}) + 3(a_{n-5})$. Assuming that $5|n$, we know that n can be written as $5k$ where k is some integer, so we may rewrite the equation to be $a_{5k} = 5(a_{5k-4}) + 3(a_{5k-5})$. Our base case is simple, we have $5|a_5 = 5$. We must show that $5|a_{5k}$ implies that 5 divides $a_{5(k+1)} = 5(a_{5(k+1)-4}) + 3(a_{5(k+1)-5})$ case, so $a_{5k+5} = 5(a_{5k+1}) + 3(a_{5k})$, but we know by assumption that a_{5k} is divisible by 5, so we have shown that $5|a_{5k} \Rightarrow 5|a_{5(k+1)}$.

Exercise 31

We can say that our equation $h_n = 4h_{n-2}$ is equivalent to $\lambda^n = 4\lambda^{n-2}$, and divide our entire equation by λ^{n-2} to get $\lambda^2 = 4$, so we know $\lambda = \pm 2$. We then have $h_n = C_1(2)^2 + C_2(-2)^n$. And we know from the statement of the problem that $h_0 = C_1 + C_2 = 0 \Rightarrow C_1 = -C_2$. We also have $h_1 = C_1(2) + C_2(-2) = 1 \Rightarrow 4C_1 = 1 \Rightarrow C_1 = 1/4 \Rightarrow C_2 = -1/4$. This implies that $h_n = \frac{(2^n)}{4} - \frac{(-2^n)}{4} = 2^{n-2} - (-2)^{n-2}$.

Exercise 39

Say we currently have the count of all possible combinations (call this count a_n) of monominos and dominoes fitting in size $1 \times n$ that satisfy the "no two consecutive dominoes" condition, for arbitrary n, and say we are tasked with finding the count of possible combinations for size $1 \times (n+1)$. Well, with the $(n+1)th$ spot we may put a monomino, and with this placement we would have the same number of possible scenarios as we did with size n (that is, a_n). We may also put a domino in this new spot, which would also bleed over into the existing spots, which means we surely couldn't have a_n again, and we also know we couldn't have a_{n-1} with this setup because a_{n-1} includes the cases where dominoes are at this closest edge which would violate the "no two consecutive dominoes" condition. So we must put a monomino after the inserted domino to be allowed to consider all possible combinations for that size. That is, we must use a_{n-2} to include the scenario of adding a domino in this new location. Therefore $a_{n+1} = a_n + a_{n-2}$ or equivalently $a_n = a_{n-1} + a_{n-3}$. We can find from calculating manually all possible scenarios that $a_3 = 3$ and $a_1 = 1$ as well as $a_2 = 2$ and using the recurrence relation we discovered we conclude that $a_0 = 1$, which gives us the initial conditions $a_0 = 1$, $a_1 = 1$, and $a_2 = 2$.

Exercise 40

Say we currently have the count of all possible scenarios (taking into account restrictions) for a string of length n and call that count a_n, now let's consider the $n+1$ case. In that new spot, we could put a 2 which would give us all the previous count (a_n), but we may also insert a 0 and then a 2 in the previous (nth) spot or a 1 and then a 2 in the nth spot, each of which have a_{n-1} possible counts, so our recurrence relation is $a_{n+1} = a_n + 2a_{n-1}$, or equivalently $a_n = a_{n-1} + 2a_{n-2}$. Let's now handle our initial conditions, a_0 and a_1. For a_1, we obviously have three choices (0,1,2) so we say a_1=3. To find a_0, we must find a_2 and use it to calculate a_0. For a_2 we may put a 0 or 1 in the first spot, and for each of those cases we can only put a 2 in the second spot, or we could put a 2 in the first spot, in which case we have 3 things we could put in the second spot. This totals $a_2 = 5$ Using our recurrence relation we find that $5 = 3 + 2(a_0)$, and solving for a_0, we get $a_0 = 1$. Now, to solve this recurrence equation we must notice its equivalence to $\lambda^n = \lambda^{n-1} + 2\lambda^{n-2}$ dividing by λ^{n-2} we get $\lambda^2 = \lambda + 2 \Rightarrow \lambda^2 - \lambda - 2 = 0$ Solving for λ using the quadratic formula (for which we spare you from the scratch work), we get $\lambda_1 = -1$ and $\lambda_2 = 2$, so $a_n = C_1(-1)^n + C_2(2)^n$. We know from our initial conditions that $a_0 = 1 = C_1 + C_2$, and we know that $a_1 = 3 = -C_1 + 2C_2$, taking the sum of these two equations $4 = 3C_2$ so $C_2 = \frac{4}{3} \Rightarrow C_1 = -\frac{1}{3}$. This implies that $a_n = -\frac{1}{3}(-1)^n + \frac{4}{3}(2)^n$.