MATH 548 Homework 2

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Section 2.7

Exercise 2

First, we consider a single ordering of the four suits of cards. In this ordering, we have 13! ways of ordering each suit. So for this configuration of suits, we have $(13!)^4$ different orderings of cards. Because we can order the suits in 4! ways, we come to the answer of $4!(13!)^4$.

Exercise 8

For each ordering of the 6 women around the table, there are 6! orderings for the men around the table. Coincidentally (simultaneously), for each ordering of the men around the table, there are 6! orderings of the women. Thus in a non-circular problem, the result would be $(6!)^2$. But due to circular ordering, we must divide by 6. Which leaves us with the answer: 6!5!.

Exercise 30

Part 1: For one parent, we could organize the children in $(5!)^2$ ways, for each of those orderings, the parent could be in any of the eleven seats. However, due to the space being circular, we must divide by the number of seats, which eliminates the multiple of eleven. Finally, we must account for the fact that the child on the right of the parent could be either a boy or a girl, which gives us a multiple of 2. So the answer is: $2(5!)^2$.

Part 2: For two parents, the answer is identical except we have $\binom{12}{2}$ locations for the two parents, and due to the circular table, we must divide by 12 instead. Which gives us the answer of:

$$\frac{\binom{12}{2} \times 2 \times (5!)^2}{12}$$

Exercise 41

Imagine first that we give the first of the children the orange, and the other two apples. That way, each child has a fruit and we have 10 more to divide up between them, without worrying about one child not having any fruit. We then approach the task of dividing the 10 remaining fruit amongst the children. This is equivalent to solving the problem $10 = A + B + C$ for $A, B, C \in \{\mathbb{N} \cup 0\}$. The solution to this smaller problem is $\binom{12}{2}$, which we must multiply by 3 to account for the fact that each of the three students may have the orange. So the result is: $3\binom{12}{2}$.

Section 3.4

Exercise 4

Let's first lay out the $2n$ numbers we are given, which should be $1, 2, 3...2n$. Let's choose all of the odd numbers from this list, which would be the set $A = \{1, 3, 5...2n-1\}$. The size of A is n, so we have yet another number to choose, but because we've already chosen all of the odd numbers, we are only left with even numbers, and because every even number in this set neighbors at least one odd number, we must have it that the difference of two of the numbers we choose is 1. I claim that the same holds easily with even numbers having been chosen.

Exercise 8

Given our rational number, $\frac{m}{n}$, we consider all of the possible remainders when m is divided by n. We know that the remainders possible are $\{0, 1, 2....n-1\}$, so, there are n possible remainders. We consider consecutive divisions, that is, long divisions of n into m from which we receive a potentially infinite string of remainders. We know that these remainders also must be in $\{0, 1, 2....n-1\}$, and we also know by basic principles of divisions that each of these $\{0, 1, 2....n-1\}$ remainders must map to one and only one other $\{0, 1, 2....n-1\}$ remainder. Because have only n choices for a remainder, and because we are considering more than n repeated divisions, we know that the remainders must repeat at least after n divisions. Because each remainder corresponds to some decimal value in the decimal expansion of the rational number, we have shown that the decimal expansion of all rational numbers must repeat itself (additionally, that it must repeat itself before n periods).o

Exercise 15

For all $a_k \in \{a_1, a_2, a_3....a_{n+1}\}$, consider $a_k \bmod n$. Because we have $n+1$ elements, by the pigeon-hole principle, we must have it that $\exists i, j < n+1$ such that $a_i \equiv a_j \bmod n$. That is, there must be at least two elements in $\{a_1, a_2, a_3....a_{n+1}\}$ that have the same remainder when divided by n. Because $a_i \equiv a_j \bmod n$ is equivalent to $n|(a_i - a_j)$, we have shown that for any $n+1$ integers $a_1, a_2, a_3....a_{n+1}$, there exists two of the integers a_i and a_j with $i \neq j$ such that $a_i - a_j$ is divisible by n.

Exercise 27

Say $A = \{1, 2, 3....n\}$ and suppose we create a one-to-one correspondence between the elements of A and the elements of a set also with n elements called B. Now suppose we populate B with n $0's$ and $1's$ in every combination possible, which means 2^n different possible values for the set B. Now let's say that each value of B corresponded to a subset of A, where the member of A corresponding to a 0 in B means that that member was not in that particular subset, and a member corresponding to a 1 in B means that that member was in the particular subset. Then we would have that the total number of subsets of A is 2^n, due to the fact that it has n elements. We now revisit our construction and consider the number of possible values of B when one of its values is fixed (this is equivalent to requiring that all subsets have at least one member in common), this is quite simply $\frac{2^n}{2} = 2^{n-1}$, so we have shown that there are at most 2^{n-1} subsets of A provided that all of the subsets have at least one element in common.